

Conferencia Internacional de Finanzas 2014

Uso de Mixturas Gaussianas en Administración de Riesgos de Mercado

Jorge Rosales Contreras
EGADE Business School
ITESM Cd de México

Agenda

- Justificación
- Mixturas Gaussianas Finitas.
- Bondad de Ajuste.
- Estimación
- Métricas de Riesgo
- Validación de Modelos

Justificación

Prueba	Variable	USDMXN	IPC	Cetes
K-S	Estadístico	0.086	0.095	0.187
	valor-p	<0.001	<0.001	<0.001
A-D	Estadístico	4.19	11.26	39.38
	valor-p	< 0.01	< 0.01	<0.01
J-B	Estadístico	2472	917	11103
	valor-p	0	0	0

Mixturas Gaussianas Finitas

$$f(\mathbf{x}; \Psi) = \sum_{i=1}^g \pi_i \phi_i(\mathbf{x}; \theta_i),$$

donde

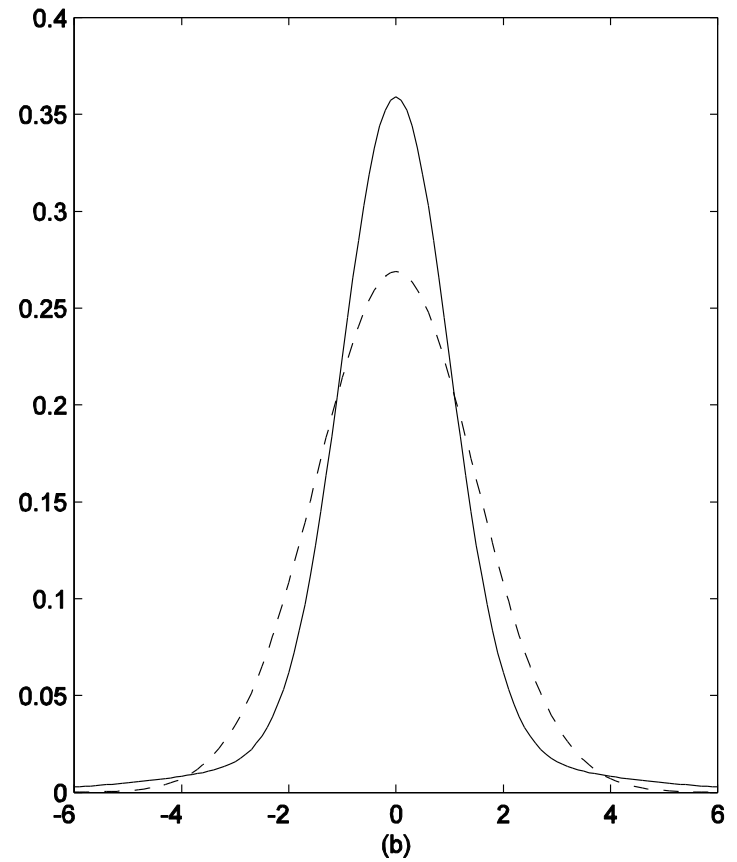
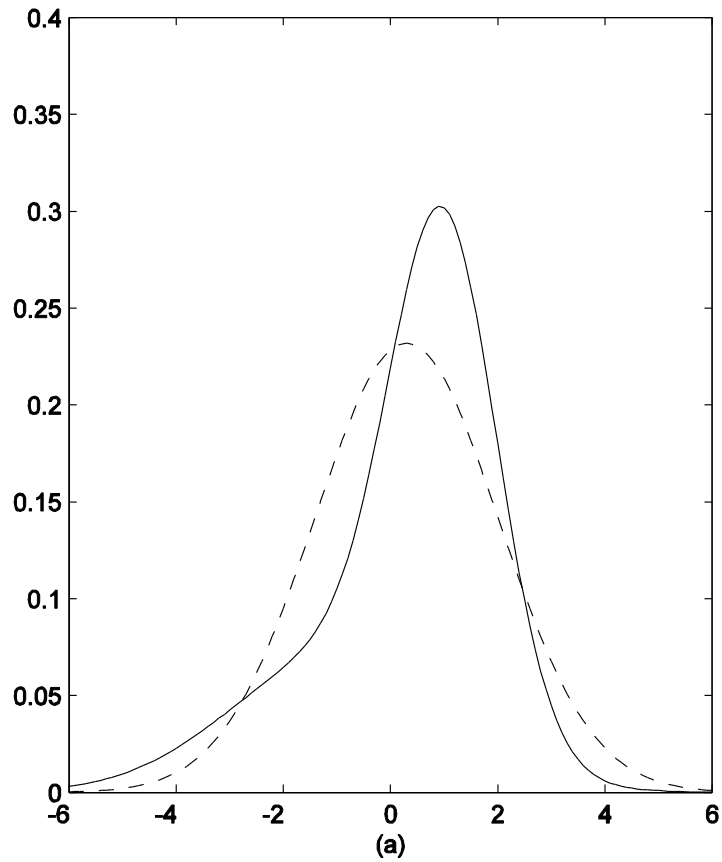
$$\phi_i(\mathbf{x}; \theta_i) = \frac{1}{(2\pi)^{d/2} |\Sigma_i|^{1/2}} \exp \left\{ -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu}_i) \Sigma_i^{-1} (\mathbf{x} - \boldsymbol{\mu}_i) \right\},$$

$$\pi_i \in (0,1), \quad \sum_{i=1}^g \pi_i = 1,$$

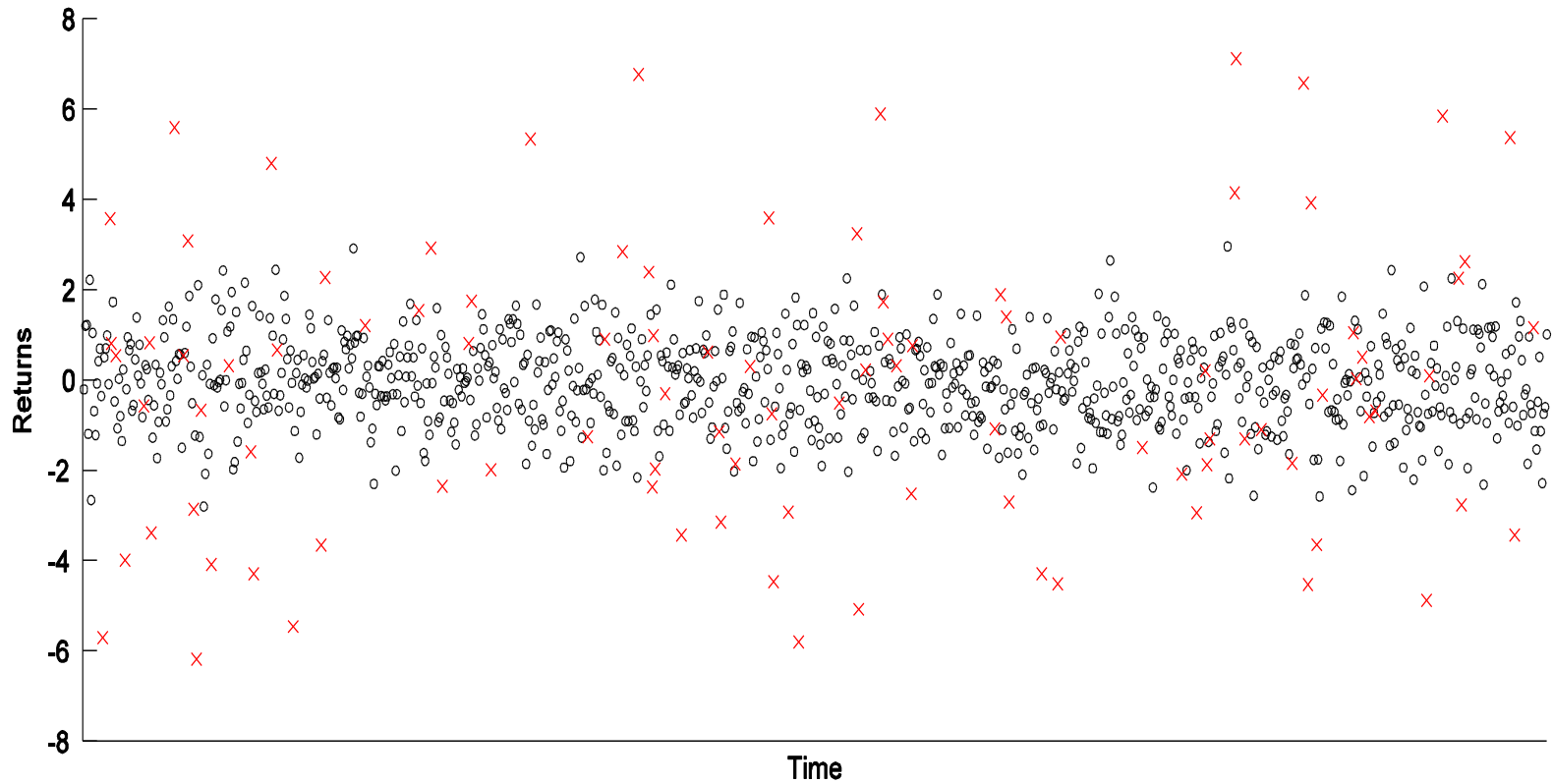
$$\theta_i = (\boldsymbol{\mu}_i, \Sigma_i^2)^t, \quad i = 1, \dots, g,$$

$$\boldsymbol{\theta} = (\boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_g)^t, \quad \Psi = (\boldsymbol{\theta}^t, \pi_1, \dots, \pi_{g-1})^t.$$

Mixturas Gaussianas Finitas



Mixturas Gaussianas Finitas



$$0.85 N(0,1) + 0.15 N(0,9)$$

Interpretación

$$Z_{1.} = 1$$
$$Y|Z_{1.} \sim \phi_1$$

Ω_1

$$Z_{2.} = 1$$
$$Y|Z_{2.} \sim \phi_2$$

Ω_2

$$Z_{g.} = 1$$
$$Y|Z_{g.} \sim \phi_g$$

Ω_g

$$Y \sim f = \sum_{i=1}^g \pi_i \phi_i$$

Interpretación

$$\Omega = \bigcup_{i=1}^g \Omega_i, \Omega_i \cap \Omega_j = \emptyset$$

$$Z_{ij} = 1 \iff Y_j(\omega) \in \Omega_i$$

Si la densidad condicional de $Y|\omega \in \Omega_i$ es ϕ_i ,
entonces $Y \sim f$.

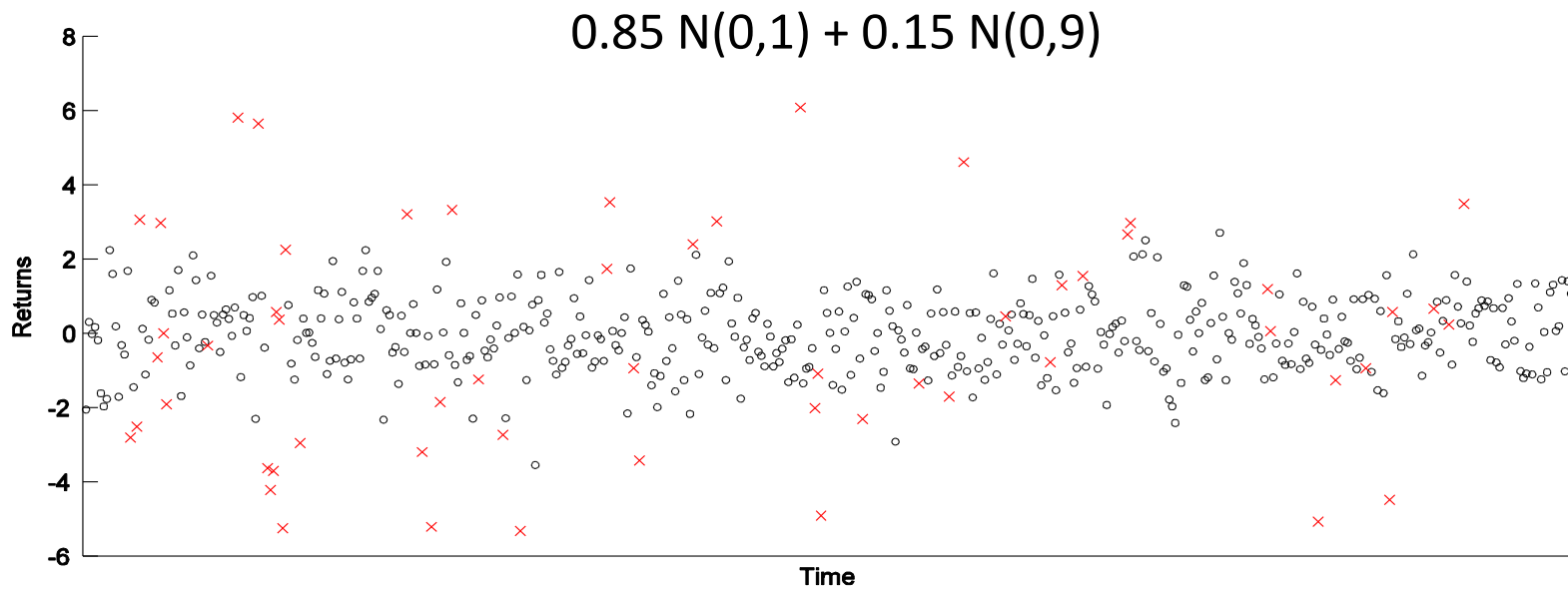
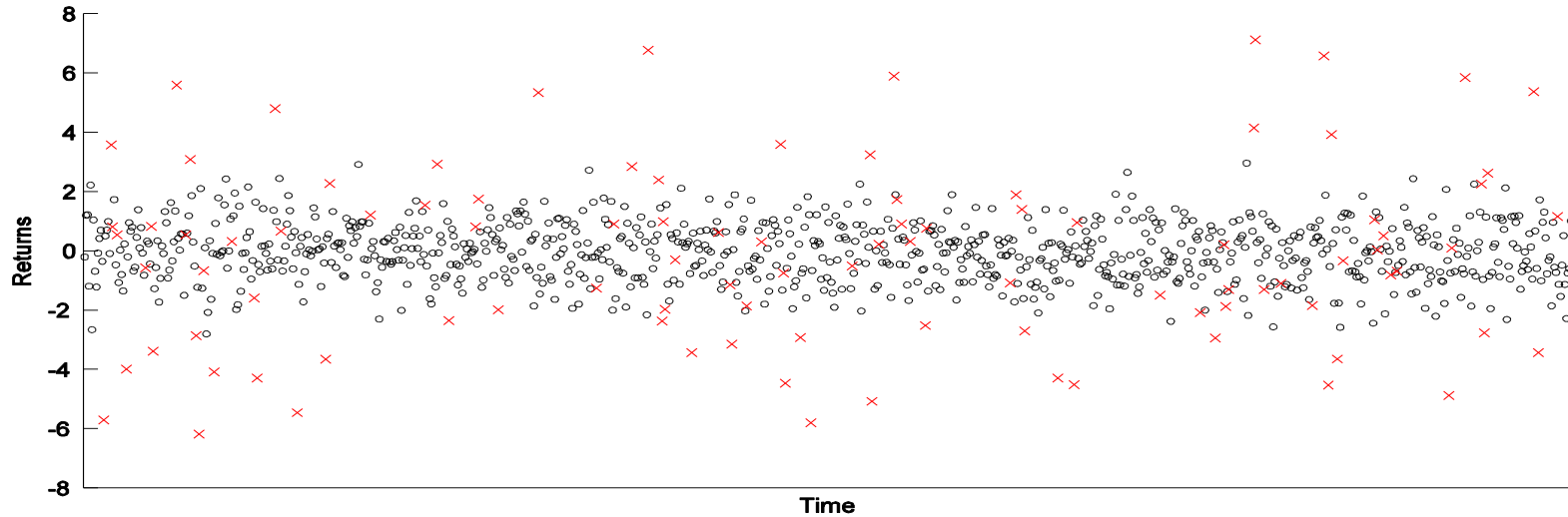
- Con $g=2$ y $\pi_2 \approx 0$, se tiene una distribución “perturbada” (Atkinson 2012).
- Cada Ω_j corresponde a un régimen de volatilidad (Alexander & Lazar 2004).

Interpretación

- $\{Z_{.j}\}_{j=1}^n$ *iid* $\Rightarrow Y_j \sim$ MG estacionaria.
- $\{Z_{.t}\}_{t=1}^T$ *Markov* $\Rightarrow Y_t \sim$ MG Markoviana oculta.

$$\rho_{Y_t}(\tau|\Psi) = (p_{11} + p_{22} - 1)^\tau \pi_1 \pi_2 \left(\frac{\mu_1 - \mu_2}{\sigma} \right)^2 .$$

Mixturas Gaussianas Finitas Markovianas



Estimación MV

$$\max_{\Psi} \sum_{j=1}^n \log \left[\sum_{i=1}^g \pi_i \phi_i(\mathbf{x}_j; \Psi) \right]$$

$$s. a. \sum_{i=1}^g \pi_i = 1.$$

Estimación MV

- Usando multiplicadores de Lagrange se obtiene un sistema de ecuaciones de la forma:

$$\hat{\pi}_{i_{MV}} = \frac{1}{n} \sum_{j=1}^n \Pr_{\Psi} \{z_{ij} = 1 | \mathbf{X} = \mathbf{x}_j\}$$

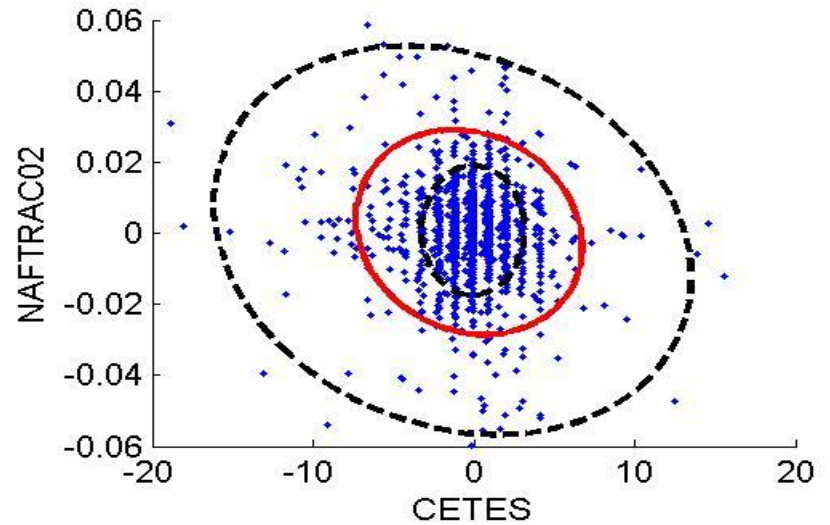
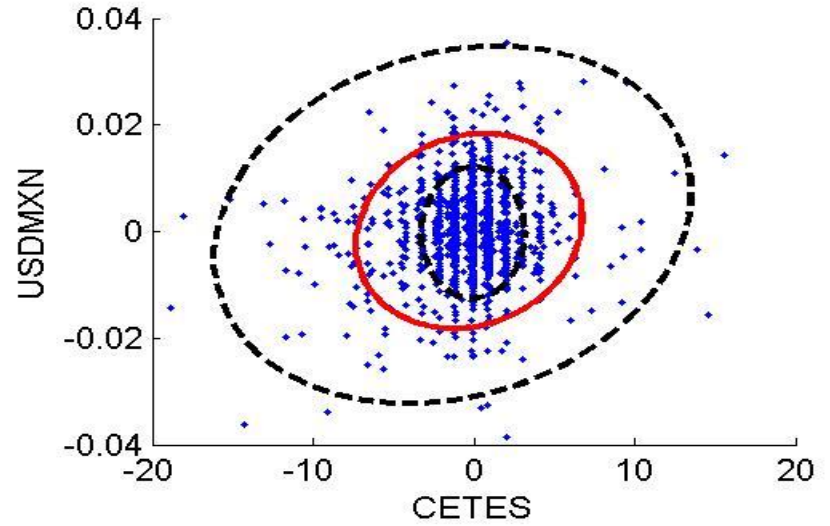
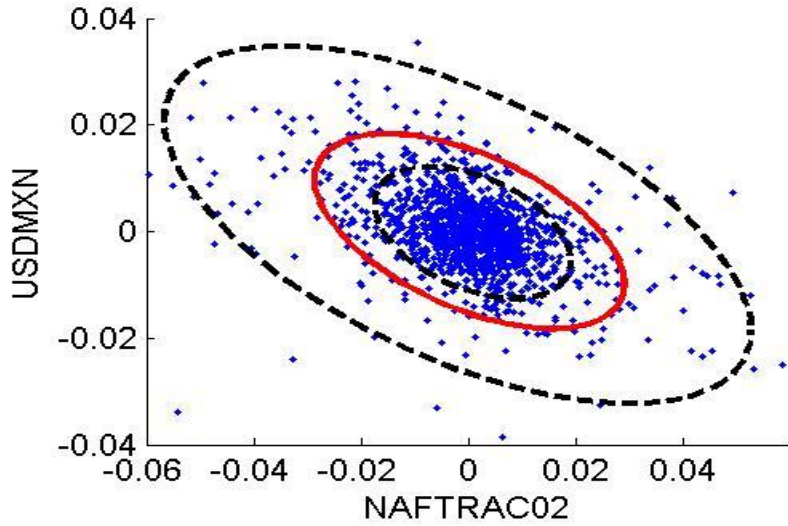
$$\sum_{j=1}^n \Pr_{\Psi} \{z_{ij} = 1 | \mathbf{X} = \mathbf{x}_j\} \frac{\partial}{\partial \boldsymbol{\theta}_i} \log \phi_i(\mathbf{x}_j; \boldsymbol{\theta}_i) = 0$$

- Estas igualdades se identifican respectivamente con los pasos Estimación y Maximización del Algoritmo EM (Dempster, Laird & Rubin 1977).

Algoritmo EM

- En la notación de McLachlan y Krishnan (2008):
- $\pi_i^{(k+1)} = \frac{1}{n} \sum_{j=1}^n \tau_i(\mathbf{x}_j, \boldsymbol{\Psi}^{(k)})$,
- $\boldsymbol{\mu}_i^{(k+1)} = \sum_{j=1}^n \omega_{ij} \mathbf{x}_j$,
- $\Sigma_i^{(k+1)} = \sum_{j=1}^n \omega_{ij} (\mathbf{x}_j - \boldsymbol{\mu}_i^{(k)}) (\mathbf{x}_j -$

Estimación: EM

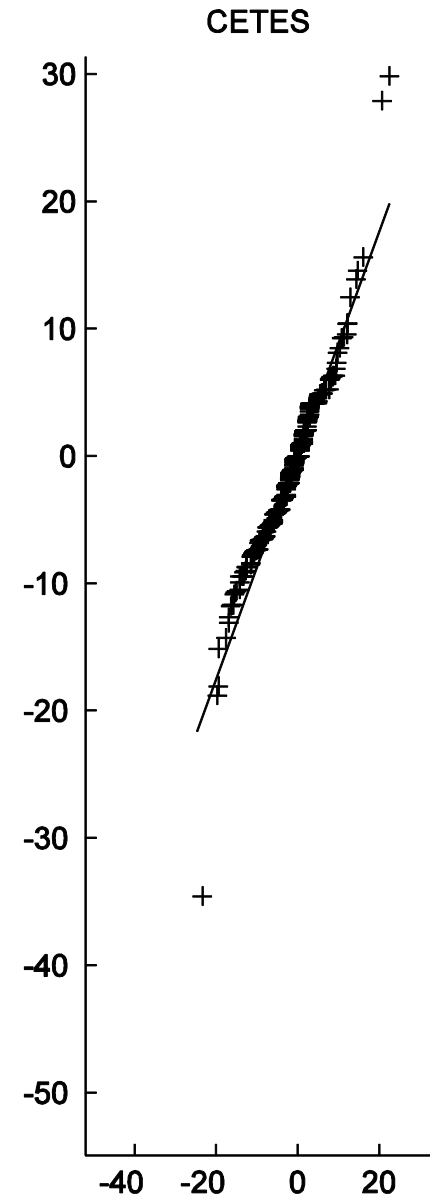
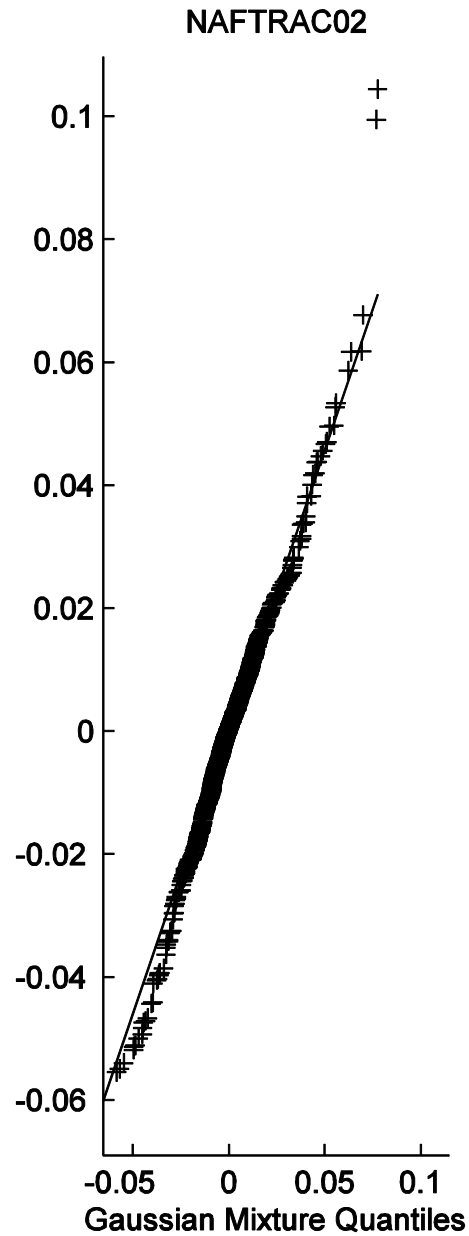
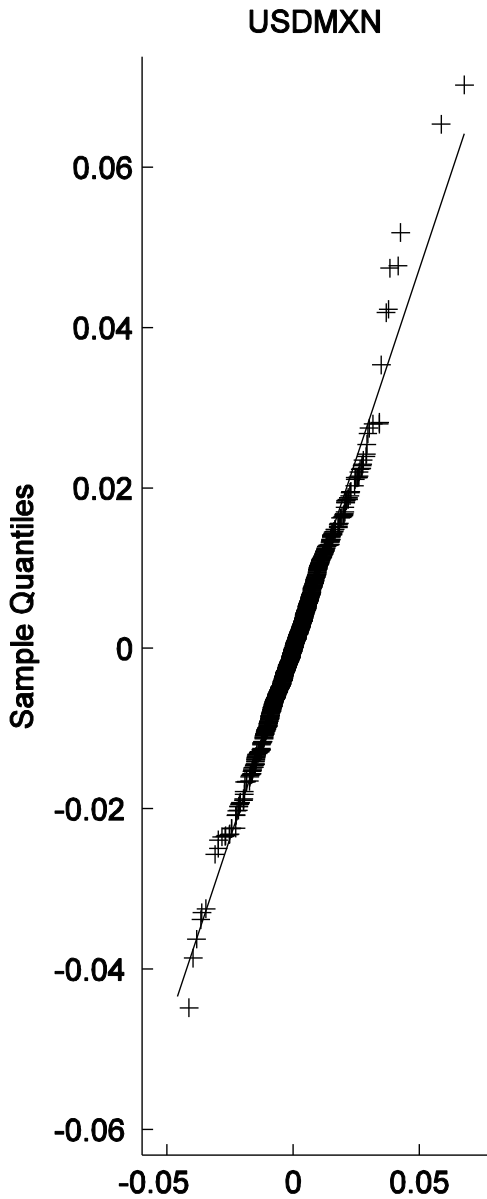


Estimación: EM

j	π_j	μ_j			Σ_j		
					0.32	-0.21	3.71
					(0.01)	(0.01)	(0.39)
1	8111	-1.30	7.82	-487		0.85	-0.31
	(114)	(0.43)	(0.41)	(19)		(0.02)	(0.51)
							26 770
							(810)
					2.33	-2.43	245
					(0.08)	(0.10)	(11.40)
2	1889	9.83	-18.84	-13 460		8.22	-419
	(114)	(1.18)	(0.21)	(705)		(0.35)	(22.75)
							541 220
							(26 920)

Estimadores y errores estándar (en paréntesis, $\times 10^{-4}$)

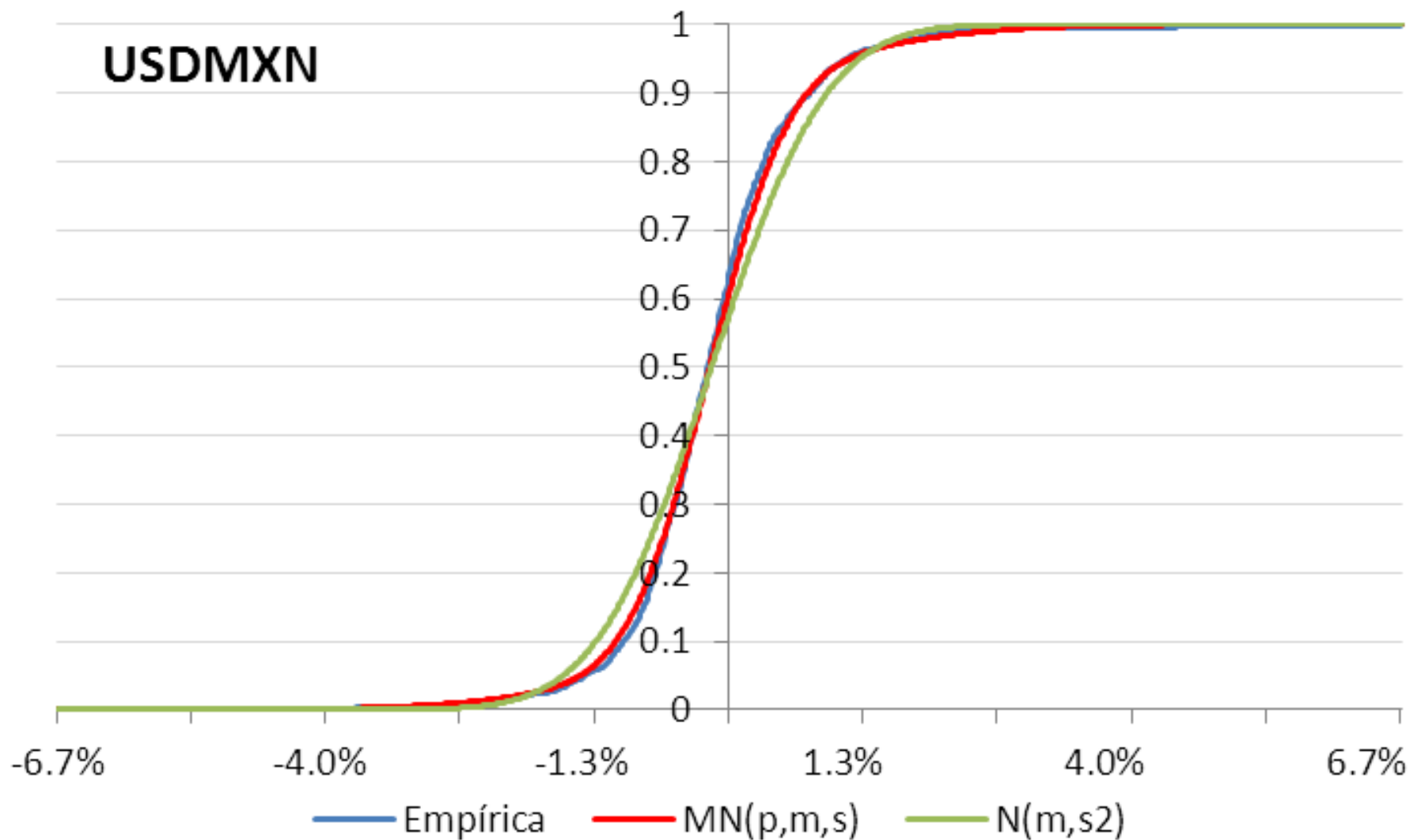
Bondad de Ajuste



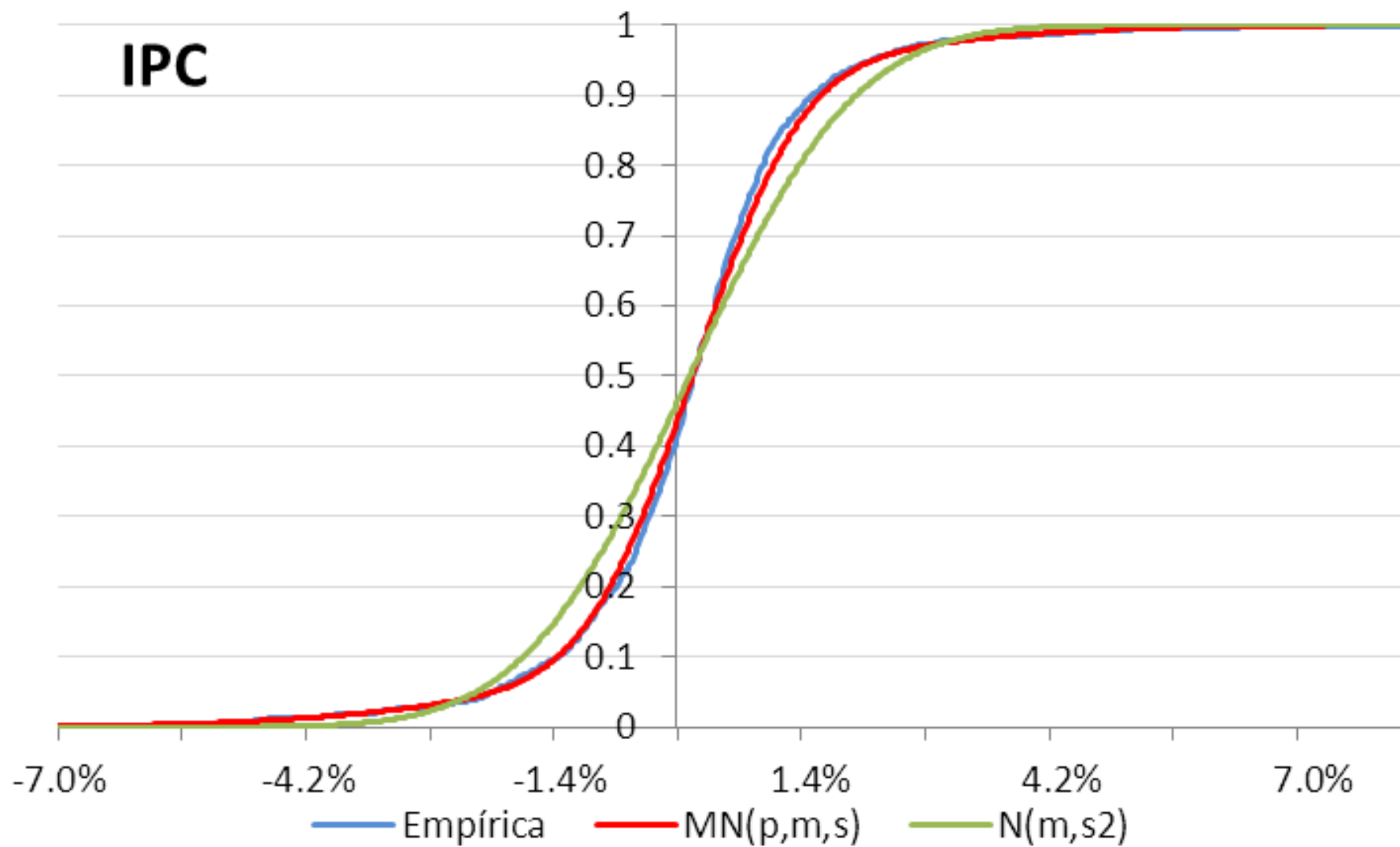
Bondad de Ajuste: K-S

Modelo	Variable	USDMXN	IPC	Cetes
Normal	K-S	0.086	0.095	0.187
	valor-p	<0.001	<0.001	<0.001
MG ₂	K-S	0.022	0.021	0.141
	valor-p	> 0.559	> 0.607	<0.001

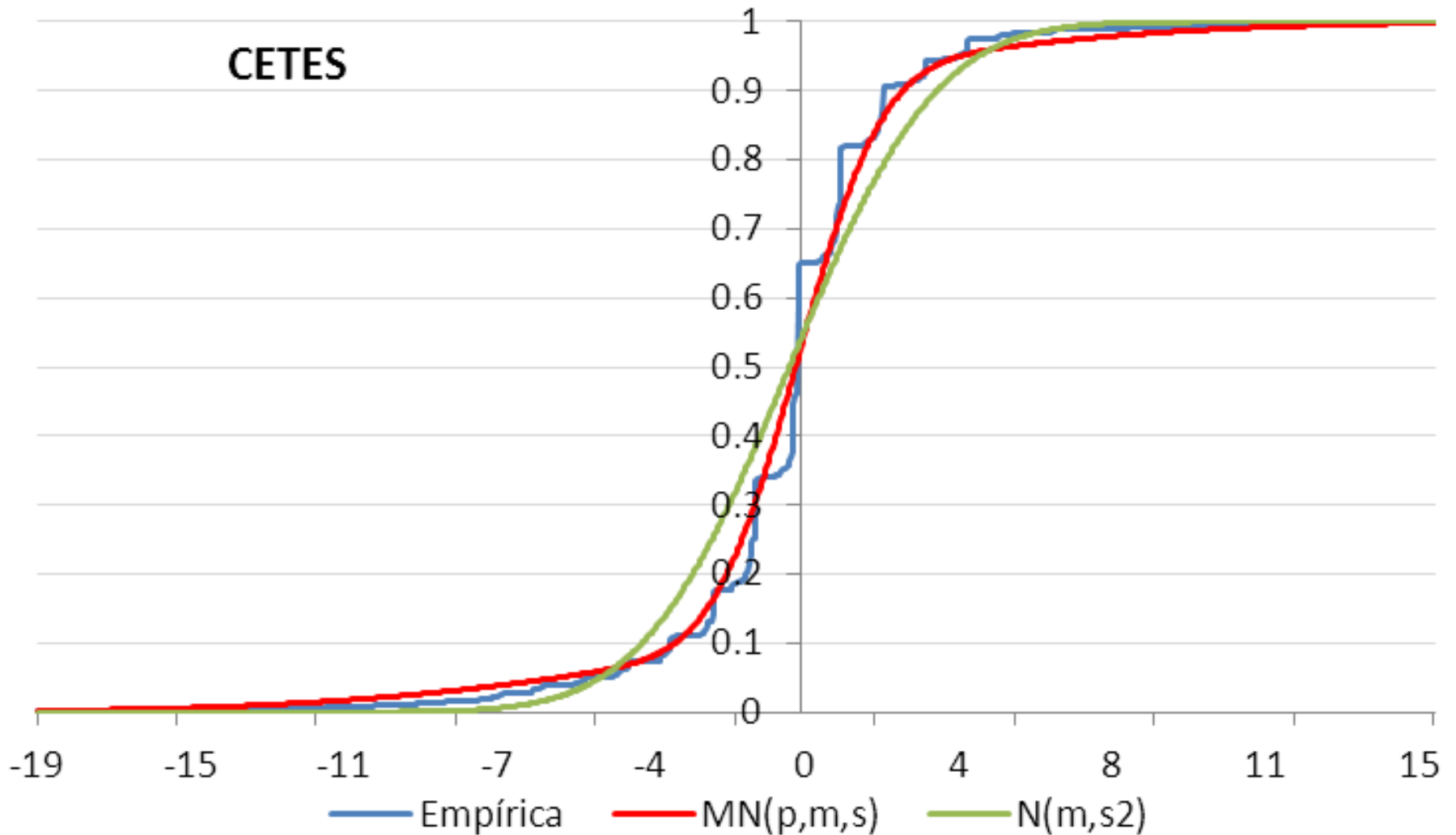
Bondad de Ajuste



Bondad de Ajuste



Bondad de Ajuste



Distribución de Pérdida

$$\begin{aligned}L_{t+1} &= -(V_{t+1} - V_t) \\ &= -[f(t+1, \mathbf{Z}_t + \mathbf{X}_{t+1}) - f(t, \mathbf{Z}_t)]\end{aligned}$$

$$\begin{aligned}l_t^\Delta(\mathbf{X}) &= -\left[f_t(t, \mathbf{Z}_t) + \sum_{i=1}^d f_{Z_i}(t, \mathbf{Z}_t) X_i \right] \\ &= -(c_t + \boldsymbol{\omega}_t' \mathbf{X})\end{aligned}$$

$$EL_{t+1} = \boldsymbol{\omega}_t' \boldsymbol{\mu} = \mu_L \quad \text{y} \quad \text{Var}(L_{t+1}) = \boldsymbol{\omega}_t' \boldsymbol{\Sigma} \boldsymbol{\omega}_t = \sigma_L^2$$

Métricas Riesgo

- Valor en Riesgo

$$VaR_{\alpha} := \inf\{u \in \mathbb{R}: F_L(u) \geq \alpha\}$$

- Déficit Esperado

$$ES_{\alpha} := \frac{1}{1 - \alpha} \int_{\alpha}^1 VaR_u du$$

– L continua con $E|L| < +\infty$:

$$ES_{\alpha} = E(L | L > VaR_{\alpha})$$

Métricas Riesgo para Mixturas Gaussianas

- VaR

$$F_L(VaR_\alpha; \Psi) = \sum_{i=1}^g \pi_i \Phi_i(VaR_\alpha; \theta_i) = \alpha$$

- Déficit Esperado.

$$ES_\alpha = \frac{1}{1-\alpha} \sum_{i=1}^g \pi_i \Phi_i(-z_{i,\alpha}) \left[\mu_i + \sigma_i \frac{\phi_i(z_{i,\alpha})}{\Phi_i(-z_{i,\alpha})} \right]$$

donde

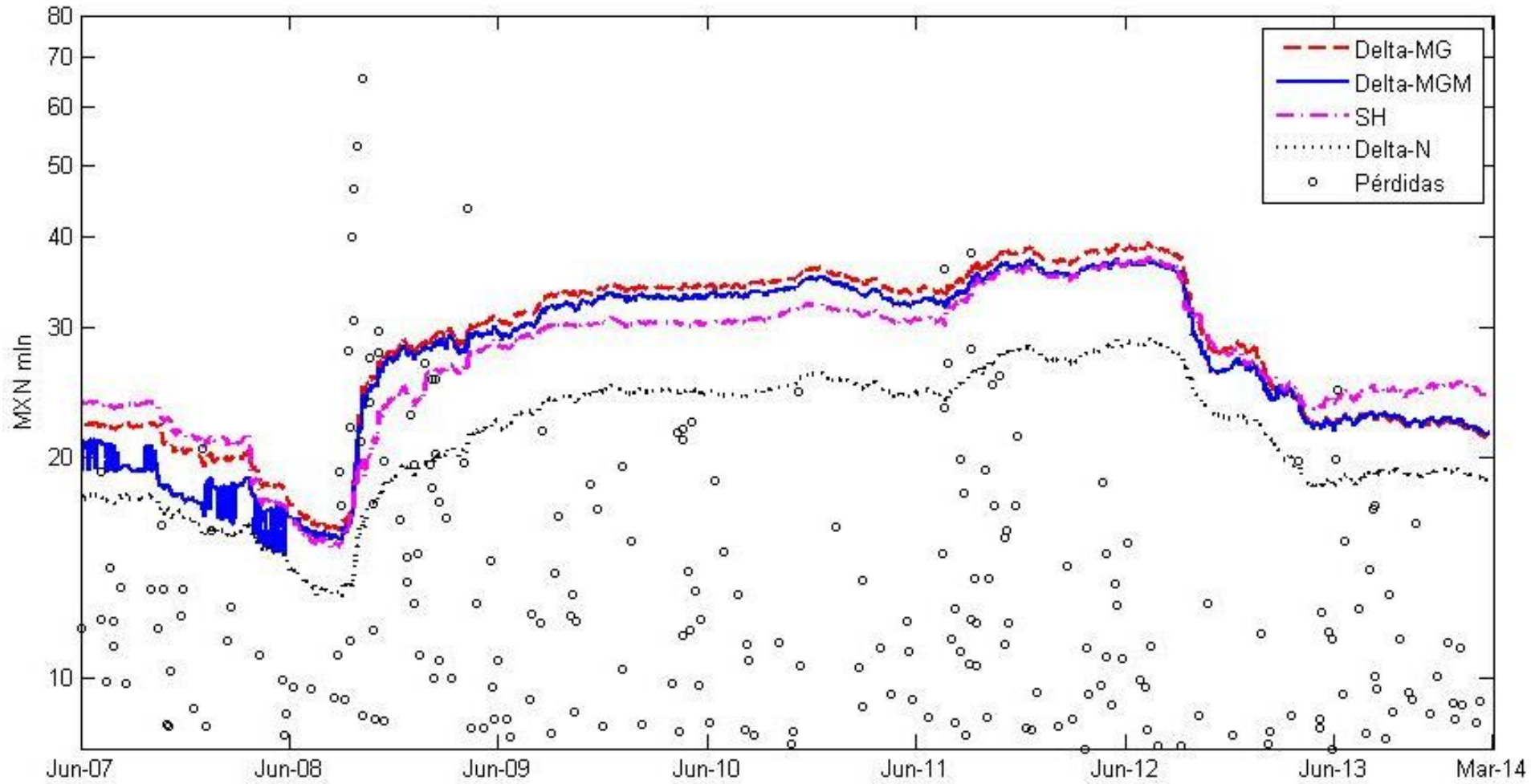
$$z_{i,\alpha} = \frac{VaR_\alpha - \mu_i}{\sigma_i}$$

Cartera de Prueba

Tipo Activo	Instrumento	Nominal o Títulos (mln)	Sensibilidad
Tipo de Cambio	USDMXN	-50	-607.0
Renta Variable	Naftrac02	10	422.6
Renta Fija	Cetes6M	15 000	-0.742

Sensibilidades al 30-abr-2013, en millones de pesos.

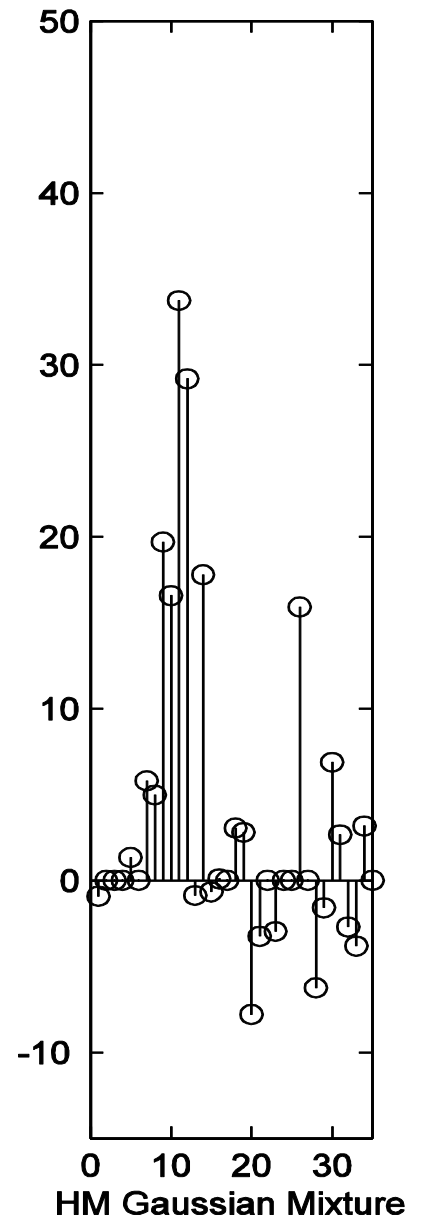
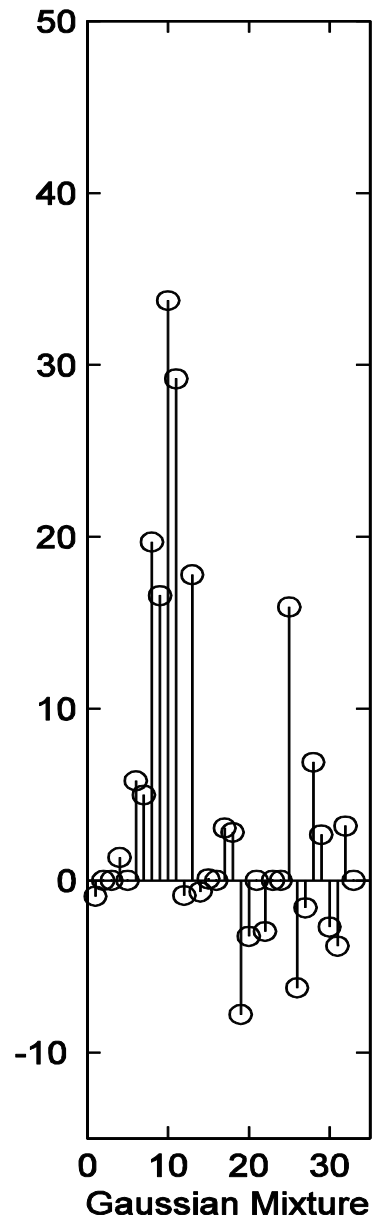
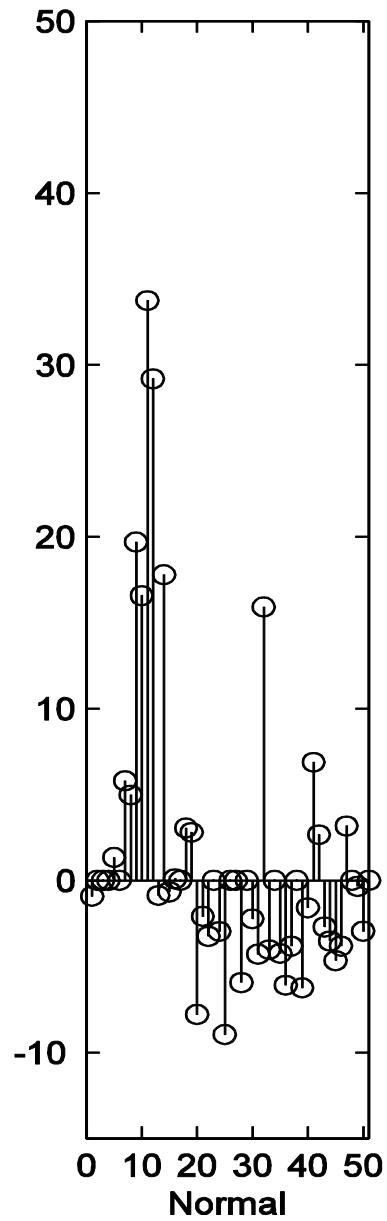
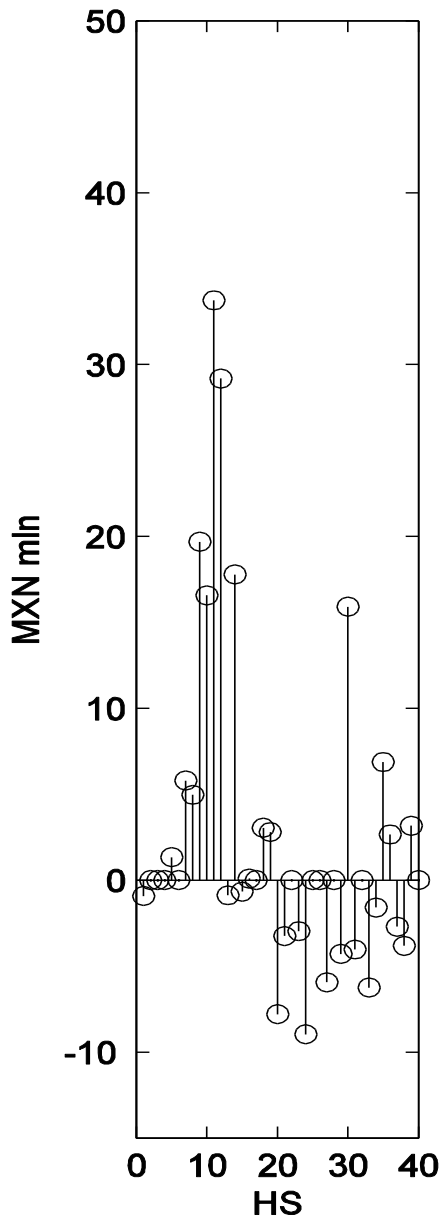
Pruebas de Validación VaR



Pruebas de Validación VaR

Confianza	Modelo	USDMXN	Naftrac	Cetes	Cartera
	SH	115	88	48	79
95%	D-N	80	87	32	77
[68,103]	D-MG	96	92	42	75
	D-MGM	93	93	42	82
	SH	54	43	25	40
97.5%	D-N	59	56	20	51
[29,58]	D-MG	44	44	18	34
	D-MGM	47	47	18	35
	SH	31	24	14	19
99%	D-N	40	37	14	31
[7,28]	D-MG	19	28	7	17
	D-MGM	19	31	7	17

Pruebas de Validación ES



Pruebas de Validación ES

Valores-p	Modelo	USDMXN	Naftrac	Cetes	Cartera
	SH	0.5212	0.6518	0.4068	0.4884
95%	D-N	0.0054	0.0008	0.0968	0.0092
	D-MG	0.5504	0.3904	0.6884	0.7782
	D-MGM	0.3410	0.1282	0.6394	0.5258
	SH	0.7674	0.1120	0.5952	0.4838
97.5%	D-N	0.0044	0.0000	0.1092	0.0066
	D-MG	0.1054	0.0112	0.3184	0.1298
	D-MGM	0.0864	0.0064	0.2630	0.1118
	SH	0.4222	0.7272	0.4918	0.4902
99%	D-N	0.0096	0.0002	0.1182	0.0126
	D-MG	0.0346	0.1618	0.1740	0.0830
	D-MGM	0.0246	0.1022	0.1522	0.0562

Referencias

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